

to occur when larger dimensions of the market,  $L$ , push researchers to increase their efforts in searching for process innovations; firstly, to avoid the decrease in the value of new patents which otherwise would be generated by smaller profits due to the higher competition and, secondly, to increase their purchasing power on a larger number of new more productive varieties.

We know that the no arbitrage condition between patents and a safe asset implies that the following Fisher equation must be satisfied for every value of  $m$

$$\frac{\pi_m}{v_m} + \frac{\dot{v}_m}{v_m} = r \quad (22)$$

We recall that while for  $m \neq i$  innovation does not introduce any new varieties, these are developed for the  $i$ -th group of firms.

## 4 Moving equilibrium

In this section we describe the properties of the equilibrium of the model, which will be characterized as a *moving equilibrium*, given that we assume that the number of firms is the slow variable of the economy, while all other variables are the fast variables.<sup>5</sup>

In particular, we know from expression (11) that in equilibrium the labor market is clearing. From (10), (6) and (17) we obtain that employment in the final sector is

$$L_C = \frac{\alpha}{w} = \frac{\alpha a}{v_i n_i} \quad (23)$$

Thus, in any periods between the two subsequent reduction in  $\gamma_i$ , the market clearing condition (11) can be rewritten as

$$L = \frac{\dot{n}_i}{n_i} a + \frac{\alpha a}{v_i n_i}$$

Let us denote with  $V_i$  the inverse of the value of the aggregate existing stock of patents of firms of type  $i$ ,  $V_i = \frac{1}{v_i n_i}$ . Then, from the previous equation, we derive the growth rate of firms of type

---

<sup>5</sup> See Schlicht (1985, 1997).

$i$  in any periods between the two subsequent reduction in  $\gamma_i$ , that is

$$g_i = \frac{L}{a} - \alpha V_i \quad (24)$$

where  $g_i = \dot{n}_i/n_i$ .

For any time before a subsequent reduction in  $\gamma_i$ , we know from expressions (22) and (23) that the rate of change of  $V_i$  for firms of type  $i$  is

$$\frac{\dot{V}_i}{V_i} = V_i \frac{n_i(1-\alpha)}{\sum_{j=1}^i n_j \left(\frac{\gamma_i}{\gamma_j}\right)^{\sigma-1}} - g_i - r \quad (25)$$

From the previous expression, we note that we have Grossman and Helpman's (1991) results only if  $\gamma_i = \gamma_j \forall j = 1, \dots, i-1$ , because this would also imply that  $n_i = n_j \forall j = 1, \dots, i-1$ . Moreover, in the same particular case, we know that the interception between the two curves (24) and (25) would be unique when  $\dot{V}_i = 0$ , as is required in equilibrium.

However, when, as it happens in our case,  $\gamma_i \neq \gamma_j$ , the intersection between the two curves (24) and (25) is not unique and it moves over time as  $n_i$  increases in any period between two following values of  $\gamma_i$  are made available. Therefore, more than a fixed steady state equilibrium, as in Grossman and Helpman (1991), our assumptions lead to identify a *moving equilibrium*, which is characterized by continuous changes in the number of firms with different productivities.

Let us define the index  $b_i$  as

$$b_i \equiv \frac{n_i \gamma_i^{1-\sigma}}{\sum_{j=1}^i n_j \gamma_j^{1-\sigma}} \quad (26)$$

It is readily verifiable that  $0 \leq b_i \leq 1$  and that  $b_i$  approaches 1 when  $n_i$  goes to infinity.  $b_i$  gives us some information on the relative weight of firms of type  $i$  on the total number of firms, where the weights are given by the productivity measure  $\gamma_i^{1-\sigma}$ . Hence, given that the value of  $b_i$  continuously changes, we have a *moving equilibrium* characterized by continuous changes in the fast variables due to movements in the slow variable  $b_i$ . Particularly, we have a moving equilibrium when all variables assume their equilibrium values conditioned to the number of patents already

introduced in the R&D sector or, in an equivalent fashion, conditioned to the value of  $b_i$ , which depends on the number of patents. Then, changes in the number of available varieties, change  $b_i$  and, consequently, other variables, as we show in the rest of this section. Expressions (24), (25) and (5) tell us that for any of those moving equilibria the following condition must be satisfied

$$\dot{V}_i = V_i \left[ V_i \left( (1 - \alpha) \frac{n_i \gamma_i^{1-\sigma}}{\sum_{j=1}^i n_j \gamma_j^{1-\sigma}} + \alpha \right) - \frac{L}{a} - \rho \right] \quad (27)$$

Moreover, we use (26) to rewrite profits (8) in the following way

$$\pi_m = \frac{(1 - \alpha)}{n_m} b_m < 1 \quad (28)$$

Substituting (26), (6) and (17) into (3), we obtain that the demand for any firm of type  $m$  is

$$x_m = \frac{n_m p_m^{1-\sigma}}{n_m p_m \sum_{j=1}^i n_j p_j^{1-\sigma}} = \frac{a \alpha b_m}{n_m \gamma_m} V_i \quad (29)$$

Using this expression we derive the total demand  $x_m n_m$  for all firms of type  $m$ , that is

$$n_m x_m = \frac{a \alpha b_m}{\gamma_m} V_i \quad (30)$$

Expression (30) tells us that when new more productive varieties are made available by the innovative sector, as the innovative process goes on, the total demand  $x_m n_m$  for the oldest firms of type  $m$ , characterized by the highest values of  $\gamma$ , tends to decrease to zero, because  $b_m$  becomes smaller and smaller. On the contrary, the total demand  $x_i n_i$  for firms of type  $i$  on the technological frontier, tends to increase as  $b_i$  increases.

Substituting (26), expression (27) becomes

$$\dot{V}_i = V_i \left\{ V_i [(1 - \alpha) b_i + \alpha] - \frac{L}{a} - \rho \right\} \quad (31)$$

Expression (31) is an upward opening parabola, with  $\dot{V}_i = 0$  either when  $V_i = 0$  or when  $V_i^* = \frac{L/a + \rho}{(1 - \alpha) b_i + \alpha} > 0$ . The graph is plotted in Figure 1 only for positive values of  $V_i$ , because negative values of  $V_i$  would have no meaning.

Insert Figure 1 about here

Moreover, in Figure 1 we also plot the actual value of  $V_i$  derived from (23), that is

$$V_i = \frac{L_C}{\alpha a} \quad (32)$$

We know from (32) that (11) and (25) are, respectively,

$$L_R = L - L_C \quad (33)$$

and

$$\dot{V}_i = \left( \frac{L_C}{\alpha a} \right) \left[ \frac{b_i L_C}{\alpha a} (1 - \alpha) - \left( \frac{L_R}{a} + \rho \right) \right] \quad (34)$$

Hence, as both pairs of equations above (31)-(32) and (33)-(34) show, the equilibrium outcomes which we describe within this framework are not stationary, given that  $b_i$  changes as the innovating process goes on determining the introduction of new varieties which increase  $n_i$ . Thus, between any pair of subsequent process innovations which lead to changes in  $\gamma_i$ , the equilibria we consider are moving equilibria which we need, indeed, to identify.<sup>6</sup>

We notice that we need to know  $L_C$  (or  $L_R$ ) in order to define the exact position of the vertical line (32) in Figure 1, otherwise, we could either have that  $L_C/(\alpha a) < V_i^*$  or that  $L_C/(\alpha a) > V_i^*$ . These two options would imply opposite changes in  $V_i$ . In fact, while  $V_i$  is increasing when  $V_i > V_i^*$ , it is decreasing when  $V_i < V_i^*$ . However, as we show in two steps,  $V_i$  must be equal to

---

<sup>6</sup> In other words, we can consider the economy as described by the following equation system in two vectors of variables  $x$  and  $y$ :

$$\dot{x} = f(x, y) \text{ and } \dot{y} = g(x, y)$$

where the vector of fast variable is  $x' = (p_m, x_m, v_m, V_m, w, \pi_R, L_R, L_C, g_i)$  and the vector of slow variable is  $y' = (b_i)$ . Note that the slow variable  $b_i$  is obtained as a transformation of the number of all variables,  $n_m$  with  $m = 1, 2, \dots, i$ , which, thus, are considered as slow variables too.

In the paper we assume that the fast vector has already reached its equilibrium for any given and fixed value of the slow variable,  $b_i$ , and we prove that the equilibrium is univocally identified for any given value of  $b_i$  in the following paragraphs in the text of the paper when we show that  $V_i = V_i^* = \frac{L/a + \rho}{(1-\alpha)b_i + \alpha}$ .

Particularly, the equilibrium value of the fast vector  $x$  is

$$x = X(b_i), \text{ with } f(X(b_i), b_i) = 0.$$

Then, given that the slow variable  $b_i$  changes over time, then  $x = X(b_i)$  gives the corresponding moving equilibrium of  $x$ . (See Schlicht (1985, 1997))

$V_i^*$ . In particular, first we recall that this is true in Grossman and Helpman's (1991, ch. 3) case.

Then we prove that this is true in our general case.

First of all, we recall that if we were in Grossman and Helpman's (1991, ch. 3) case,  $\gamma$  would assume only one value, that is  $\gamma_m = \gamma_i = \gamma_1 \forall m$ . Moreover, in this case the steady state equilibrium is characterized by  $\dot{V}_1 = 0$  and  $L_R = L(1 - \alpha) - a\alpha\rho$ . In fact, we know that in this case the expectations of agents are fulfilled only if the economy jumps immediately to the point in which  $\dot{V}_1 = 0$ , because if  $\dot{V}_1$  were positive we would have  $V_1$  growing to infinity, while if  $\dot{V}_1$  were negative, we would end up with  $V_1 = 0$ . However, Grossman and Helpman (1991) show that both cases are impossible, given that: in the first case we cannot have  $V_1$  growing to infinity because  $L_R$  would be drawn to zero,  $n_1$  would stop growing, and  $v_1$  would be different from zero (given that with a finite number of variety, profits are strictly positive); in the second case, we cannot have  $V_1 = 0$ , because  $L_R$  would assume its maximum potential value,  $L$ , with  $L_C = 0$ , and expectations would be contradicted.<sup>7</sup> Finally, we notice that in this case,  $b_1 = 1$ . If we consider the pair of equations (31)-(32) which describes the equilibrium condition, they would intersect in  $V_1 = \frac{L-L_R}{\alpha a} = V_1^* = L/a + \rho$  with  $L_R = L(1 - \alpha) - a\alpha\rho$  derived from the second pair of equations (33)-(34).

Let us now consider the case in which, in the framework so far described, an innovation process takes place producing new patents characterized by  $\gamma_2 < \gamma_1$ , which perturbs previous stationary equilibrium.<sup>8</sup> These new patents allow  $n_2$  firms to employ the technology of type 2. We know from (21) that  $n_2 = \frac{1}{a}n_1L_R$  and that  $b_2 = \frac{n_2\gamma_2^{1-\sigma}}{n_1\gamma_1^{1-\sigma} + n_2\gamma_2^{1-\sigma}} < 1$ .

---

<sup>7</sup> In fact, Grossman and Helpman (1991, p. 61) recall that if  $L_R = L$ , the number of varieties would grow continuously and, at the same time we would have  $v_1(t) = \int_t^\infty e^{-r(\tau-t)} \frac{1-\alpha}{n_1(t)} d\tau < \frac{1-\alpha}{rn_1(t)}$ .

In fact,  $v_1(t) = \int_t^\infty e^{-r(\tau-t)} \frac{1-\alpha}{n_1(t)} d\tau = \left[ -\frac{e^{-r(\tau-t)}}{r} \right]_t^\infty \frac{1-\alpha}{n_1(t)} = \left[ -\frac{e^{-r\infty}}{r} + \frac{1}{r} \right] \frac{1-\alpha}{n_1(t)} < \frac{1-\alpha}{rn_1(t)}$ .

Therefore, we would have that  $v_1(t)n_1(t) < \frac{1-\alpha}{r}$  which is equivalent to saying  $V_1(t) > \frac{r}{1-\alpha} > 0$  which contradicts the fact that  $V_1 = 0$ .

<sup>8</sup> In the particular example described by (20) when  $n_1L = \chi_1$ . In any case, we recall that we do not need to use this particular specification of the more general expression (19).

After the change in  $\gamma$ , the innovative sector continues to produce new patents of type 2 according to (12). The inverse of the aggregate value of patents of type 1 is equal to  $V_1 = \frac{1}{v_1 n_1}$ , where  $n_1$  is now a constant. Moreover, from (28), we know that profits of firms of type 1 are from now on  $\pi_1 = \frac{(1-\alpha)}{n_1} b_1$ . At the same time, there will be continuous increases in  $n_2$ , or in other more productive types of firms whenever there are further innovations leading to further reductions in  $\gamma$ . These processes will reduce  $b_1$ , reducing profits of firms of type 1 and, therefore, the value of patents of firms of type 1,  $v_1$ , thereby, increasing  $V_1$ . Therefore, we know that  $V_1$  is increasing in  $b_1$ . Moreover, for any given value of  $b_1$ ,  $n_1$  and  $v_1$  are given and, thus,  $V_1$  is univocally determined. In other words, we are able to rule out bubble paths for the aggregate value of firms which are no longer on the technological frontier, such as firms of type 1, once technology with  $\gamma_2$  can be used. Furthermore, we may say that this is generally true for any firms of type  $m$  different from  $i$ , that is, firms at the technological frontier from the production process point of view, because their number  $n_m$  does not increase anymore, and because their value  $v_m$  must decrease due to the ongoing growth in variety. At the limit, when the weight  $b_m$  of firms adopting older technology than the firms at the frontier,  $\gamma_i$  tends to decrease toward zero, and  $V_m$  tends to infinity.

**Lemma 1** *For any variety which is not at the technological frontier, that is, for any variety produced with  $\gamma_m > \gamma_i$ , profits decrease and  $V_m$  increases as the weight  $b_m$  of the group decreases as a consequence of subsequent innovations in the R&D sector which increase the number of patents.*

Moreover, returning to our example, we further observe that once the new patents of type 2 become available at the technological frontier, with  $i = 2$ , and  $b_2 < 1$ , then  $V_2^* = \frac{L/a+\rho}{(1-\alpha)b_2+\alpha} > V_1^*$ .

Then we notice that while firms of type 2 remain at the frontier, for a given value of  $b_2$ , if  $V_2 = \frac{1}{v_2 n_2}$  does not immediately jump to  $V_2^*$ , there could be two other possible cases which we should consider: either we have  $V_2 < V_2^*$  (with  $\dot{V}_2 < 0$  which would draw  $V_2$  to zero), or  $V_2 > V_2^*$  (with  $\dot{V}_2 > 0$  and  $V_2$  growing to infinity). We note, in passing, that the following arguments can be generalized to the case in which firms of type  $i$  are at the frontier, for given  $b_i$  values.

We rule out the first case, that is  $V_2 < V_2^*$ , because we want to exclude asset bubble paths

both in the subcase in which all firms of type 2 will always continue to be on the technological frontier in the future and in the subcase in which, sometime in the future, these firms will no longer be on the technological frontier due to further process innovations, which further reduce  $\gamma$  for future varieties. In the first subcase,  $V_2$  cannot be drawn to zero because this would be possible, for a finite number of firms  $n_2$ , only with  $v_2$  increasing to infinity; but with ongoing patent innovations this is impossible. Following the same reasoning, we can exclude asset bubble paths in the present (while firms of type 2 are at the frontier) also in the subcase in which the same firms were no longer supposed to be on the technological frontier at sometime in the future. In fact, if in the future the number of varieties is growing, we can exclude a continuous growth in  $v_2$  and consequent decreases in  $V_2$  in the present, for given value of  $n_2$  and  $b_2$ , because profits of firms of type 2 are superiorly limited by variety growth.

We can also rule out the second case, that is  $V_2 > V_2^*$ , once more both in the subcase in which all firms of type 2 will always continue to be on the technological frontier in the future and in the subcase in which, at sometime in the future, these firms will no longer be on the technological frontier due to further process innovations. In the first subcase, when firms of type 2 remain on the frontier with  $i = 2$ , this will rapidly lead to employ all workers in the manufacturing sector with no more growth in variety (given that we know from (32) that  $V_2 = \frac{L_C}{\alpha a}$ ) and  $V_2$  increasing to infinity. However, this would be possible only if  $v_2$  were equal to zero and we exclude this case because if the number of varieties stops growing, profits must always be strictly positive. Finally, in the second subcase if  $n_2$  stops growing at sometime in the future because firms of type 2 are displaced from the frontier, in the present (while these firms are at the frontier), we know that  $V_2$  will increase to infinity in the future, because it is in the future that  $v_2$  tends to zero as  $b_2$  (and profits) tends to zero. However, these future increases in  $V_2$  cannot be anticipated in the present, because, otherwise,  $V_2$  would be lead to infinity in the present which would be inconsistent with  $L_C < L$ . In other words, from (32) we could have it only with  $L_C = L$ , but at the expense of no innovation at all in the present (because  $L_R = 0$ ) which would exclude the potential future

process innovations.

So we have that when firms of type two are at the technological frontier,  $V_2$  immediately jumps to  $V_2^*$  and that, for any given value of  $b_2$ ,  $\dot{V}_2 = 0$ . We know that  $b_2$  changes in the present with ongoing patent innovations. In particular, it continuously increases, because  $n_2$  increases. As long as firms of type two are at the technological frontier, the continuous increase in  $b_2$  is associated with continuous reductions in  $V_2 = V_2^*$ , with  $V_2$  having a lower limit. In fact,  $V_2 = V_2^*$  tends to  $V_1^*$  as  $n_2$  increases.

We already noticed that previous arguments can be generalized to the case in which, instead of firms of type 2 we consider firms of type  $i$  at the frontier for given  $b_i$  values. For we know now that we have

$$V_i = V_i^* = \frac{L/a + \rho}{(1 - \alpha)b_i + \alpha} \quad (35)$$

where  $V_i = V_i^*$  is increasing in  $L$ ,  $\rho$  (with  $\rho = r$ ) and decreasing in  $a$ ,  $b_i$  and  $\alpha$ .

We may write the following lemma:

**Lemma 2** *In general, the inverse of the aggregate value of firms at the frontier,  $V_i$ , is  $V_i = V_i^* > V_1^*$  with  $\dot{V}_i = 0$  when  $b_i$  is given. However, given that  $b_i$  increases when patent innovations take place for firms at the technological frontier increasing  $n_i$ ,  $V_i$  changes approaching  $V_1^*$  as the weight  $b_i$  increases.*

The previous lemma is extremely important given that it allows us to identify not a unique steady state equilibrium, but a series of moving equilibria, which can be considered as perturbations of the original steady state equilibrium in Grossman and Helpman (1991, ch. 3), and which continue to change as long as  $b_i$  changes and as long as we can have different process innovations which continue to change the technology at the frontier. The implications of our results will be discussed in the following section.

Moreover, from expressions (30) and (35) we obtain the total demand for varieties of type  $i$

$$x_i n_i = \frac{a\alpha}{\gamma_i} \left[ \frac{L/a + \rho}{(1 - \alpha) + \alpha/b_i} \right]$$

which clearly shows that as  $b_i$  increases, total demand for varieties of type  $i$  increases. Thus, gradually in our model, while the market share and the demand of previously developed varieties



decreases, the market share and the demand of new and more productive varieties made available increases as long as they are on the technological frontier.

## 5 Structural changes and the scale effect

One of the most striking characteristics of the moving equilibrium we have so far described is that it allows us to represent the effects of ongoing patent innovations which take place together with process innovations. Considering both kinds of innovations gives a more complete picture of the effects of R&D activities and it produces a setup in which the rate of growth of patent innovations varies across time according to workers' distribution between the final and the innovative sectors considered in the model.

In the period in which technology of type  $i$  is available, we know from expression (12) that the rate of innovation is proportional to the number of workers employed in the R&D sector, and this number  $L_R$ , derived from (33)-(34) when  $\dot{V}_i = 0$ , depends on the value of  $b_i$ , that is

$$L_R = \frac{Lb_i(1-\alpha) - a\rho\alpha}{(1-\alpha)b_i + \alpha} \quad (36)$$

As in Grossman and Helpman (1991), we assume that  $L$  is sufficiently large to allow patent innovations to take place: this requires that  $L > a\rho\alpha/b_i(1-\alpha)$ . Once more, it is readily verifiable that when  $b_i = 1$  we obtain the same results as in Grossman and Helpman (1991).

Expression (36) shows that the number of workers employed in the innovative sector is an increasing function of  $b_i$  because

$$\frac{\partial L_R}{\partial b_i} = \frac{(1-\alpha)\alpha(L+a\rho)}{((1-\alpha)b_i + \alpha)^2} > 0$$

Therefore, when there are at least two different types of firms producing using different technologies, and the innovative sector intensifies its research in finding new patents for the production of new goods employing the more productive technologies, then any time a new patent is produced and implemented the value of  $b_i$  increases. As  $b_i$  increases, the final sector in aggregate

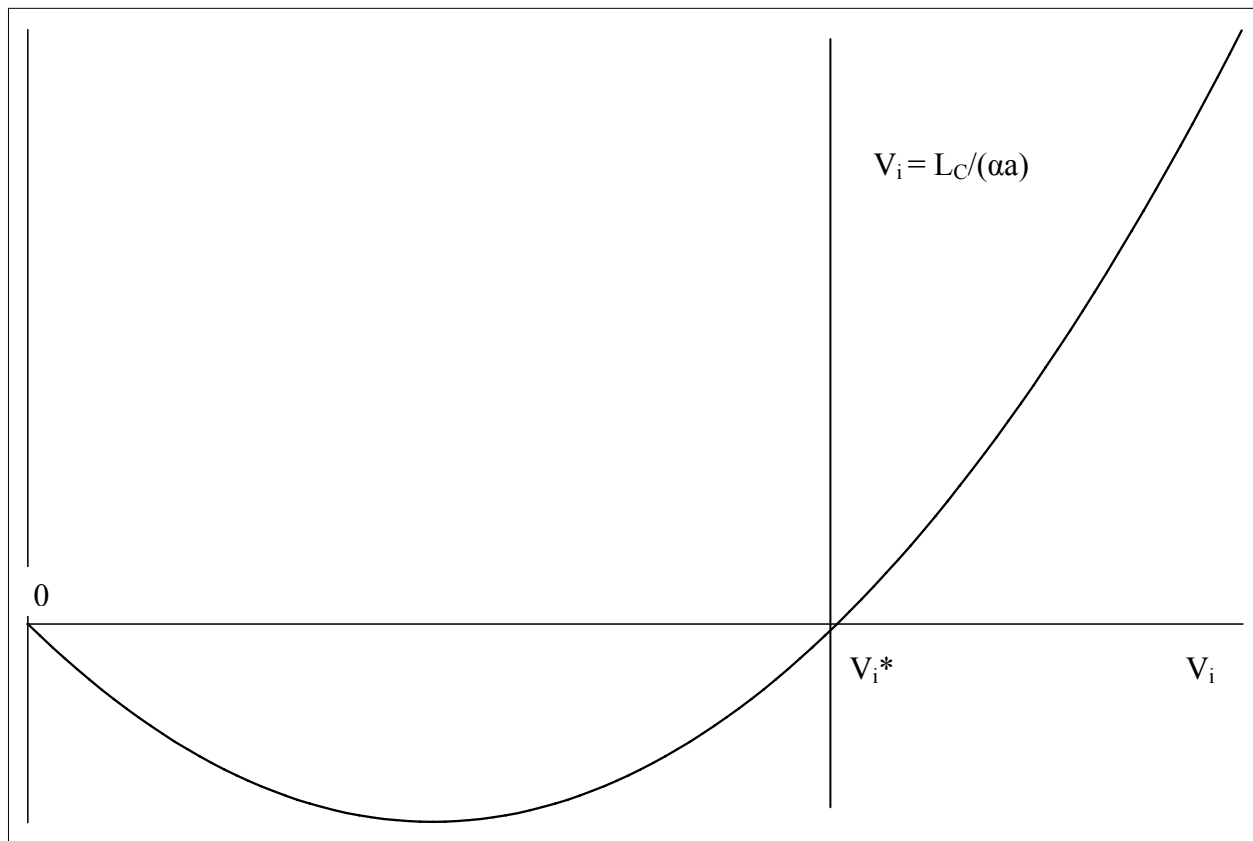


Figure 1